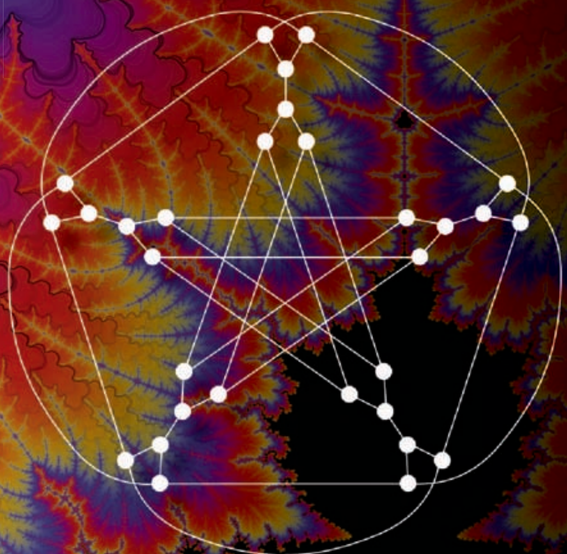
TEXTBOOKS IN MATHEMATICS

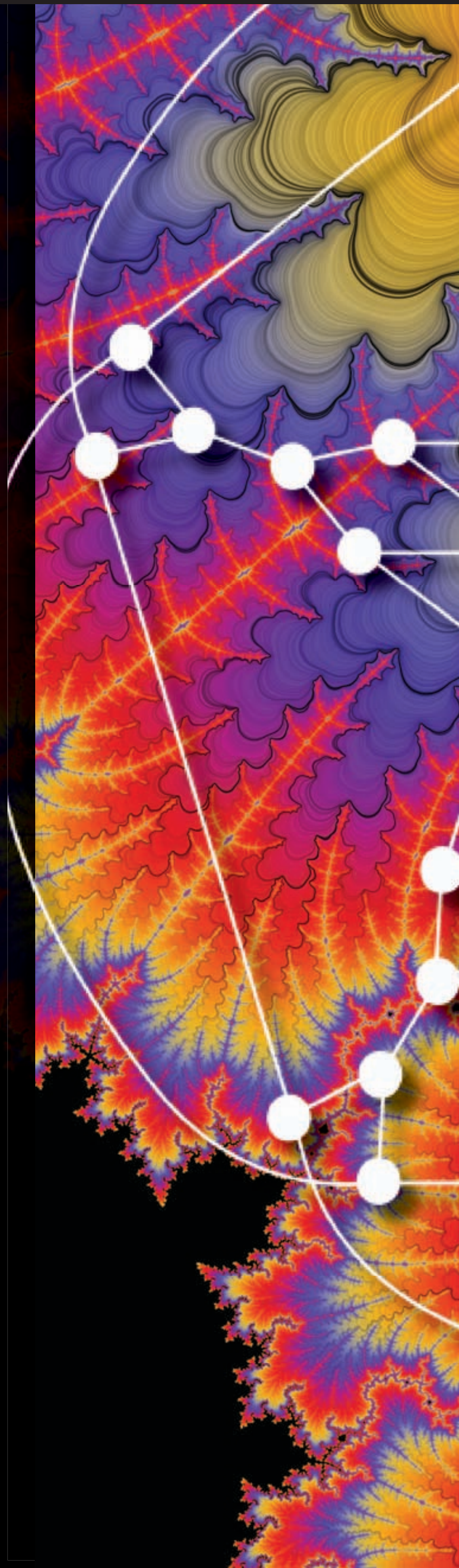
GRAPHS & DIGRAPHS

SIXTH EDITION



GARY CHARTRAND
LINDA LESNIAK
PING ZHANG

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK



TEXTBOOKS in MATHEMATICS

GRAPHS & DIGRAPHS

SIXTH EDITION

GARY CHARTRAND
Western Michigan University
Kalamazoo, USA

LINDA LESNIAK
Western Michigan University
Kalamazoo, USA

PING ZHANG
Western Michigan University
Kalamazoo, USA

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

TEXTBOOKS in MATHEMATICS

Series Editors: Al Boggess and Ken Rosen

PUBLISHED TITLES

ABSTRACT ALGEBRA: AN INQUIRY-BASED APPROACH
Jonathan K. Hodge, Steven Schlicker, and Ted Sundstrom

ADVANCED LINEAR ALGEBRA
Nicholas Loehr

ADVANCED LINEAR ALGEBRA
Hugo Woerdeman

ADVANCED LINEAR ALGEBRA, SECOND EDITION
Bruce N. Cooperstein

APPLIED ABSTRACT ALGEBRA WITH MAPLE™ AND MATLAB®, THIRD EDITION
Richard Klima, Neil Sigmon, and Ernest Stitzinger

APPLIED DIFFERENTIAL EQUATIONS: THE PRIMARY COURSE
Vladimir Dobrushkin

COMPUTATIONAL MATHEMATICS: MODELS, METHODS, AND ANALYSIS WITH MATLAB® AND MPI, SECOND EDITION
Robert E. White

DIFFERENTIAL EQUATIONS: THEORY, TECHNIQUE, AND PRACTICE, SECOND EDITION
Steven G. Krantz

DIFFERENTIAL EQUATIONS: THEORY, TECHNIQUE, AND PRACTICE WITH BOUNDARY VALUE PROBLEMS
Steven G. Krantz

DIFFERENTIAL EQUATIONS WITH MATLAB®: EXPLORATION, APPLICATIONS, AND THEORY
Mark A. McKibben and Micah D. Webster

ELEMENTARY NUMBER THEORY
James S. Kraft and Lawrence C. Washington

EXPLORING LINEAR ALGEBRA: LABS AND PROJECTS WITH MATHEMATICA®
Crista Arangala

GRAPHS & DIGRAPHS, SIXTH EDITION
Gary Chartrand, Linda Lesniak, and Ping Zhang

PUBLISHED TITLES CONTINUED

INTRODUCTION TO ABSTRACT ALGEBRA, SECOND EDITION
Jonathan D. H. Smith

INTRODUCTION TO MATHEMATICAL PROOFS: A TRANSITION TO ADVANCED MATHEMATICS, SECOND EDITION
Charles E. Roberts, Jr.

INTRODUCTION TO NUMBER THEORY, SECOND EDITION
Marty Erickson, Anthony Vazzana, and David Garth

LINEAR ALGEBRA, GEOMETRY AND TRANSFORMATION
Bruce Solomon

MATHEMATICAL MODELLING WITH CASE STUDIES: USING MAPLE™ AND MATLAB®, THIRD EDITION
B. Barnes and G. R. Fulford

MATHEMATICS IN GAMES, SPORTS, AND GAMBLING–THE GAMES PEOPLE PLAY, SECOND EDITION
Ronald J. Gould

THE MATHEMATICS OF GAMES: AN INTRODUCTION TO PROBABILITY
David G. Taylor

MEASURE THEORY AND FINE PROPERTIES OF FUNCTIONS, REVISED EDITION
Lawrence C. Evans and Ronald F. Gariepy

NUMERICAL ANALYSIS FOR ENGINEERS: METHODS AND APPLICATIONS, SECOND EDITION
Bilal Ayyub and Richard H. McCuen

ORDINARY DIFFERENTIAL EQUATIONS: AN INTRODUCTION TO THE FUNDAMENTALS
Kenneth B. Howell

RISK ANALYSIS IN ENGINEERING AND ECONOMICS, SECOND EDITION
Bilal M. Ayyub

TRANSFORMATIONAL PLANE GEOMETRY
Ronald N. Umble and Zhigang Han

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20151109

International Standard Book Number-13: 978-1-4987-3580-3 (eBook - PDF)

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

To

the memory of my mother and father. G. C.

my mother and the memory of my father Stanley. L. L.

my mother and the memory of my father. P. Z.

Table of Contents

Preface to the Sixth Edition

Graph theory is an area of mathematics whose origin dates back to 1736 with the solution of the famous Königsberg Bridge Problem by the eminent Swiss mathematician Leonhard Euler. During the next several decades, topics in graph theory arose primarily through recreational mathematics. The development of graph theory received a substantial boost in 1852 when the young British mathematician Francis Guthrie introduced one of the best known problems in all of mathematics: the Four Color Problem. It wasn't until late in the 19th century, however, when graph theory became a theoretical area of mathematics through the research of the Danish mathematician Julius Petersen. Major progress in graph theory, however, didn't occur until World War II ended. Since then, though, the subject has developed into an area with a fascinating history, numerous interesting problems and applications in many diverse fields. It is the beauty of the subject, however, that has attracted so many to this field.

The goal of this sixth edition is, as with the previous editions, to describe much of the story that is graph theory – through its concepts, its theorems, its applications and its history. The audience for the sixth edition is beginning graduate students and advanced undergraduate students. The primary prerequisite required of students using this book is a knowledge of mathematical proofs. For some topics, an elementary knowledge of linear algebra and group theory is useful. For Chapter 21, an elementary knowledge of probability is needed. Proofs of some of the results that appear in this book have not been supplied because the techniques are beyond the scope of the book or are inordinately lengthy. Nevertheless, these results have been included due to their interest and since they provide a more complete description of what is known on a particular topic.

A one-semester course in graph theory using this textbook can be designed by selecting topics of greatest interest to the instructor and students. There is more than ample material available for a two-semester sequence in graph theory. Our goal has been to prepare a book that is interesting, carefully written, student-friendly and consisting of clear proofs. The sixth edition has been divided into shorter chapters as well as more sections and subsections to make reading and locating material easier for instructors and students. The following major additions have been made to the sixth edition:

- more than 160 new exercises
- several conjectures and open problems
- many new theorems and examples
- new material on graph decompositions
- a proof of the Perfect Graph Theorem

- material on Hamiltonian extension
- a new chapter on the probabilistic method in graph theory and random graphs.

At the end of the book is an index of mathematicians, an index of mathematical terms and an index of symbols. The references list research papers referred to in the book (indicating the page number(s) where the reference occurs) and some useful supplemental references. There is also a section giving hints and solutions to all odd-numbered exercises.

Over the years, there have been some changes in notation that a number of mathematicians now use. When certain notation appears to have been adopted by sufficiently many mathematicians working in graph theory so that this has become the norm, we have adhered to these changes. As with the fifth edition, the following notation is used in the sixth edition:

- a path is now expressed as $P = (v_1, v_2, \ldots, v_k)$ and a cycle as $C = (v_1, v_2, \ldots, v_k, v_1)$;
- the Cartesian product of two graphs G and H is expressed as $G \,\square\, H$, rather than the previous $G \times H$;
- the union of G and H is expressed by $G + H$, rather than $G \cup H$;
- the join of two graphs G and H is expressed as $G \vee H$, rather than $G + H$.

We are most grateful to Bob Ross, senior editor of CRC Press, who has been a constant source of support and assistance throughout the entire writing process.

Gary Chartrand, Linda Lesniak and Ping Zhang

Chapter 1

Introduction

The theory of graphs is one of the few fields of mathematics with a definite birth date.

It is the subject of graph theory of course that we are about to describe. The statement above was made in 1963 by the mathematician Oystein Ore who will be encountered in Chapter 6. While graph theory was probably Ore's major mathematical area of interest during the latter part of his career, he is also known for his work and interest in number theory (the study of integers) and the history of mathematics.

Although awareness of integers can be traced back for many centuries, geometry has an even longer history. Early geometry concerned distance, lengths, angles, areas and volumes, which were used for surveying, construction and astronomy. While geometry dealt with magnitudes, the German mathematician Gottfried Leibniz introduced another branch of geometry called the geometry of position. This branch of geometry did not deal with measurements and calculations, but rather with the determination of position and its properties. The famous mathematician Leonhard Euler said that it hadn't been determined what kinds of problems could be studied with the aid of the geometry of position but in 1736 he believed that he had found one, which led to the origin of graph theory. It is this event to which Oystein Ore was referring in his quote above. We will visit Euler again, in Chapter 5 as well as in Chapters 10 and 11.

1.1 Graphs

Graphs arise in many different settings. Let's look at three of these.

Example 1.1 Eight students $s_1, s_2, \ldots, s_8$ have been invited to a dinner. Each student knows only some of the other students. The students that each student knows are listed below.

$s_1: s_4, s_5, s_8$	$s_2: s_3, s_4, s_6, s_8$
$s_3: s_2, s_5, s_6, s_7, s_8$	$s_4: s_1, s_2, s_5, s_6, s_7$
$s_5: s_1, s_3, s_4, s_8$	$s_6: s_2, s_3, s_4, s_7$
$s_7: s_3, s_4, s_6$	$s_8: s_1, s_2, s_3, s_5$

In order to determine if these eight students can be seated at a round table where each student sits next to two students he or she knows, it is useful to represent this situation by the diagram shown in Figure 1.1. Each point or small circle in the diagram represents a student and two points are joined by a line segment if the two students know each other. This diagram is referred to as a *graph*.

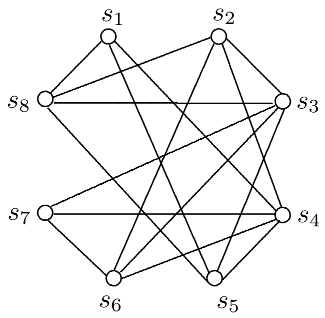


Figure 1.1: The diagram in Example 1.1

A related question is whether the students could be seated at a round table so that each student sits next to two students he or she does not know. ♦

Example 1.2 There are six special locations in a neighborhood park. Twelve trails are to be built between certain pairs of these locations, namely all pairs of locations except $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$ (see Figure 1.2(a)). A trail can be straight or curved. Can this be done without any trails crossing? This situation can be represented by the diagram with six points (each point representing a location), where two points are joined by a line segment or a curve if the two points represent locations to be joined by a trail (see Figure 1.2(b)). Once again, this diagram is a graph. ♦

Example 1.3 A chemical company is to ship eight chemicals (denoted by $c_1, c_2, \ldots, c_8$) to a chemistry department in a university. Because some pairs of chemicals should not be shipped in the same container, more than one container needs to be used for this shipment. Each chemical is listed below together with the chemicals that should not be placed in the same container as this chemical.

$c_1: c_2$	$c_2: c_1, c_8$	$c_3: c_5, c_6, c_7$	$c_4: c_5, c_7$
$c_5: c_3, c_4, c_8$	$c_6: c_3, c_7$	$c_7: c_3, c_4, c_6$	$c_8: c_2, c_5$

It would be useful to know the minimum number of containers needed to ship these eight chemicals. This situation can be represented by the diagram

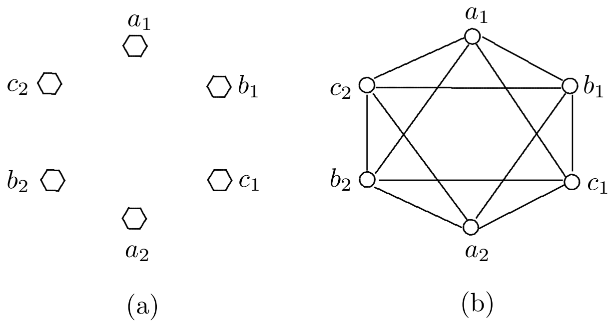


Figure 1.2: Constructing a graph in Example 1.2

in Figure 1.3, whose eight points represent the eight chemicals and where two points are joined by a line segment or curve if these chemicals cannot be shipped in the same container. Here too, this diagram is a graph. ♦

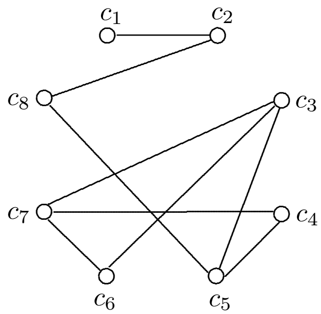


Figure 1.3: The graph in Example 1.3

We now give a formal definition of the term *graph*. A **graph** G is a finite nonempty set V of objects called **vertices** (the singular is **vertex**) together with a possibly empty set E of 2-element subsets of V called **edges**. Vertices are sometimes referred to as **points** or **nodes**, while edges are sometimes called **lines** or **links**. In fact, historically, graphs were referred to as *linkages* by some. Calling these structures graphs was evidently the idea of James Joseph Sylvester (1814–1897), a well-known British mathematician who became the first mathematics professor at Johns Hopkins University in Baltimore and who founded and became editor-in-chief of the first mathematics journal in the United States (the *American Journal of Mathematics*).

To indicate that a graph G has **vertex set** V and **edge set** E, we write $G = (V, E)$. To emphasize that V and E are the vertex set and edge set of a graph G, we often write V as $V(G)$ and E as $E(G)$. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. If $e = uv$ is an edge of G, then e is said to **join** u and v.

As the examples described above indicate, a graph G can be represented by a diagram, where each vertex of G is represented by a point or small circle and an edge joining two vertices is represented by a line segment or curve joining the corresponding points in the diagram. It is customary to refer to such a diagram as the graph G itself. In addition, the points in the diagram are referred to as the vertices of G and the line segments are referred to as the edges of G. For example, the graph G with vertex set $V(G) = \{u, v, w, x, y\}$ and edge set $E(G) = \{uv, uy, vx, vy, wy, xy\}$ is shown in Figure 1.4. Even though the edges vx and wy cross in Figure 1.4, their point of intersection is not a vertex of G.

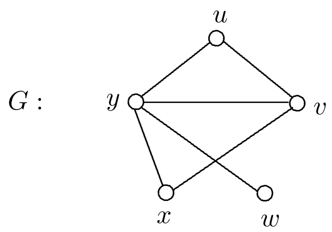


Figure 1.4: A graph

If uv is an edge of G, then u and v are **adjacent vertices**. Two adjacent vertices are referred to as **neighbors** of each other. The set of neighbors of a vertex v is called the **open neighborhood** of v (or simply the **neighborhood** of v) and is denoted by $N_G(v)$, or $N(v)$ if the graph G is understood. The set $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. If uv and vw are distinct edges in G, then uv and vw are **adjacent edges**. The vertex u and the edge uv are said to be **incident** with each other. Similarly, v and uv are incident.

For the graph G of Figure 1.4, the vertices u and v are therefore adjacent in G, while the vertices u and x are not adjacent. The edges uv and vx are adjacent in G, while the edges vx and wy are not adjacent. The vertex v is incident with the edge uv but is not incident with the edge wy.

The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. The order of the graph G of Figure 1.4 is 5 and its size is 6. We typically use n and m for the order and size, respectively, of a graph. A graph of order 1 is called a **trivial graph**. A **nontrivial graph** therefore has two or more vertices. A graph of size 0 is called an **empty graph**. A **nonempty graph** then has one or more edges. In any empty graph, no two vertices are adjacent. At the other extreme is a **complete graph** in which every two distinct vertices are adjacent. The size of a complete graph of order n is $\binom{n}{2} = n(n-1)/2$. Therefore, for every graph G of order n and size m, it follows that $0 \le m \le \binom{n}{2}$. The complete graph of order n is denoted by K_n. The complete graphs K_n for $1 \le n \le 5$ are shown in Figure 1.5.

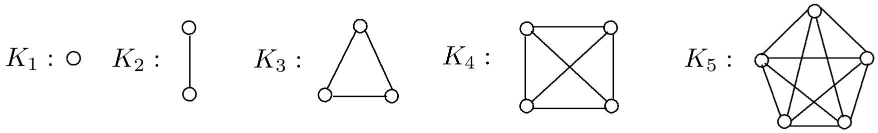


Figure 1.5: Some complete graphs

Two other classes of graphs that are often encountered are the paths and cycles. For an integer $n \geq 1$, the **path** P_n is a graph of order n and size $n-1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. For an integer $n \geq 3$, the **cycle** C_n is a graph of order n and size n whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. The cycle C_n is also referred to as an **n-cycle** and the **3-cycle** is also called a **triangle**. The paths and cycles of order 5 or less are shown in Figure 1.6. Observe that $P_1 = K_1$, $P_2 = K_2$ and $C_3 = K_3$.

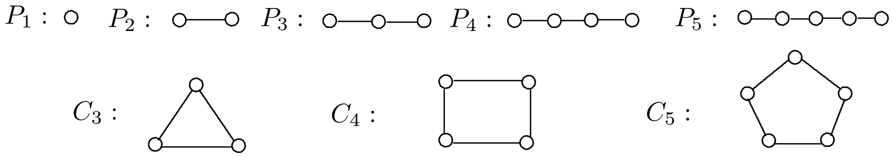


Figure 1.6: Paths and cycles of order 5 or less

1.2 The Degree of a Vertex

The **degree of a vertex** v in a graph G is the number of vertices in G that are adjacent to v. Thus, the degree of v is the number of vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of edges incident with v. The degree of a vertex v is denoted by $\deg_G v$ or, more simply, by $\deg v$ if the graph G under discussion is clear. Hence, $\deg v = |N(v)|$. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. An edge incident with an end-vertex is called a **pendant edge**. The largest degree among the vertices of G is called the **maximum degree** of G and is denoted by $\Delta(G)$. The **minimum degree** of G is denoted by $\delta(G)$. (The symbols Δ and δ are the upper case and lower case Greek letter delta, respectively.) Thus, if v is a vertex of a graph G of order n, then

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n-1.$$

For the graph G of Figure 1.4,

$$\deg w = 1,\ \deg u = \deg x = 2,\ \deg v = 3 \text{ and } \deg y = 4.$$

Thus, $\delta(G) = 1$ and $\Delta(G) = 4$.